

Holographic Entropy Packing inside a Black Hole

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If general relativity is spontaneously induced, the black hole limit is governed by a phase transition which occurs precisely at the would have been horizon. The exterior Schwarzschild solution then connects with a novel core of vanishing spatial volume. The Kruskal structure, admitting the exact Hawking imaginary time periodicity, is recovered, with the conic defect defused at the origin, rather than at the horizon. The entropy stored inside any interior sphere is universal, equal to a quarter of its surface area, thus locally saturating the 't Hooft-Susskind holographic bound. The associated Komar mass and material energy functions are non-singular.

Black hole thermodynamics is anchored to the well known Bekenstein-Hawking area entropy formula [1]. Intriguingly, neither the Gibbons-Hawking [2] Euclidean path integral derivation, nor the more locally oriented Wald's [3] derivation, make use of the black hole interior. This seems to tell us that aside from the singularity in its origin, the interior region of a black hole is quite a 'boring' place. Following this line, the area entropy has triggered the fascinating idea that, from some obscure reason, no physical degrees of freedom can reside within the interior of a black hole. If it is the case, these degrees of freedom, whatever they are, are owed to live on or near the horizon surface, with one bit of information per a quarter of Planck area [4]. The brane paradigm model, which introduces the notion of a stretched horizon [5], is a realization of this idea. The catch is, however, that the horizon looks perfectly innocent to in-falling matter. The apparent inconsistency between the horizon as a physical entity and as the mere point of no return has ignited a famous debate in the physical society. The black hole entropy formula has also inspired the so-called holographic principle [6]. The latter, primarily introduced by 't Hooft [7], attempting to resolve the black hole information paradox, and further developed by Susskind [8] to deal with black hole complementarity, is recently gaining theoretical support from the AdS/CFT duality [9].

Apparently, as far as entropy packing is concerned, the interior of a black hole seems to be superfluous, certainly within the framework of general relativity per se. However, if general relativity is not a fundamental theory, but rather a spontaneously induced theory of gravity, the black hole limit has been shown [10] to be governed by a phase transition which occurs precisely at the would have been horizon. The fully recovered general relativistic exterior solution then connects with a novel core of vanishing spatial volume. The idea of horizon phase transition [11] is not new, and so is the notion of black stars and fuzzballs [12]. In this paper, after reviewing the fine details of the underlying phase transition geometry [10], we show how the black hole entropy is consistently packed inside the whole interior region. This is done in a universal manner, while locally saturating the 't Hooft-Susskind holographic bound, and is formulated by eqs.(15, 19).

A simple theory which allows for a spontaneously induced general relativity, is given by the action

$$I = \int \left(-\frac{\varphi^2 \mathcal{R}}{16\pi} - \frac{1}{16\pi a} \left(\varphi^2 - \frac{1}{G} \right)^2 + \mathcal{L}_m \right) \sqrt{-g} d^4x . \quad (1)$$

The role of the Higgs potential is to allow the conformally coupled Brans-Dicke [13] scalar field $\varphi(x)$ to acquire its vacuum expectation value $\langle \varphi \rangle = G^{-1/2}$ by virtue of Zee mechanism [14], and in full analogy with the electro/weak interactions. Note that the theory is fully equivalent to $f(R) = R + \frac{1}{4}aR^2$ gravity, with stability á la Sotiriou-Faraoni [15] guaranteed for $a > 0$. The value of a can be made as small as necessary to be compatible with Solar System tests. An arbitrary scalar kinetic term can always be added without affecting our main conclusions, but even in the absence of such a term, the scalar field $\varphi(x)$ is dynamical. The fact that the matter Lagrangian \mathcal{L}_m does not couple to the scalar field defines the Jordan, rather than Einstein, frame to be the physical one (our main results are nevertheless frame independent). For the sake of the present paper, our interest lies with static spherically symmetric vacuum solutions, with the standard line element taking the form

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2 . \quad (2)$$

The general relativistic $\varphi(r) = G^{-1/2}$ Schwarzschild solution, which we hereby tag with some $\epsilon = 0$ (related to the scalar charge), is accompanied by a more general class of asymptotically flat $\epsilon \neq 0$ solutions. The metric associated with the latter solution has been recently derived and analyzed [10]. Some details of the analysis are crucial for the sake of the present paper, so a brief review is in order.

The large distance expansion of the generic solution is quite conventional, see ref.([10]), with a scalar charge Q_s factorizing a variety of Yukawa suppressed terms at a typical length scale \sqrt{aG} , namely

$$\begin{aligned} e^{\nu(r)} &\simeq 1 - \frac{2GM}{r} + \frac{Q_s}{\sqrt{aG}} N(r) \frac{e^{-r/\sqrt{aG}}}{r^{GM/\sqrt{aG}}} , \\ e^{-\lambda(r)} &\simeq 1 - \frac{2GM}{r} + \frac{Q_s}{\sqrt{aG}} L(r) \frac{e^{-r/\sqrt{aG}}}{r^{GM/\sqrt{aG}}} , \end{aligned} \quad (3)$$

$$G\varphi^2(r) \simeq 1 + \frac{Q_s}{r} F(r) \frac{e^{-r/\sqrt{aG}}}{r^{GM/\sqrt{aG}}},$$

where $F(r), N(r), L(r) = 1 + \mathcal{O}(r^{-1})$. Clearly, for $Q_s = 0$, the Schwarzschild solution is fully recovered, not just asymptotically. However, with the focus on the $Q_s \rightarrow +0$ limit, one can run a full scale numerical solution, starting from the above asymptotic expressions, and already notice the appearance of a phase transition near the would have been horizon, at $r \simeq 2GM$. In other words, *representing a 'level crossing' effect (one parameter family degenerates), the limit $Q_s \rightarrow 0$ does not reproduce the $Q_s = 0$ solution.* In fact, this phase transition has already been verified analytically [10], with the derived transition profile, which connects the Schwarzschild exterior with the novel interior in the vicinity of $r \simeq 2GM$, given by

$$e^{\nu(r)} \simeq \frac{\epsilon}{6} e^{-\sigma(r)} \left(\frac{6}{\epsilon} - e^{\sigma(r)} \right), \quad e^{\lambda(r)} \simeq e^{2\sigma(r)} e^{\nu(r)}, \quad (4)$$

$$r - 2GM \simeq 2GM \left(e^{-\sigma(r)} + \frac{\epsilon}{6} \log \left(\frac{6}{\epsilon} e^{-\sigma(r)} - 1 \right) \right),$$

with $e^{\sigma(r)}$ serving as a parametric function. It is quite possible that the limit $\epsilon \rightarrow 0$ is physically unattainable, and that one has to settle for a tiny yet finite ϵ , such that the invariant width of the transition profile eq.(5) is of order Planck length, highly reminding us of the 'stretched horizon' [5]. The transition profile is insensitive to the terms involving the scalar potential, and exhibits a remarkable self similarity feature, as expressed by the fact that $\epsilon \rightarrow k\epsilon$ only causes scale changes $e^\lambda \rightarrow k^{-1}e^\lambda$ and $r - 2GM \rightarrow k(r - 2GM)$. The exact relation $\epsilon(Q_s)$ is generally quite complicated, and at this stage, was only obtained numerically by plotting $\frac{1}{2}r(\nu' + \lambda')$. The only analytic exception being the large \sqrt{aG} case, for which

$$\epsilon \simeq \frac{3Q_s}{2GM}. \quad (5)$$

The forthcoming analysis, however, does not depend on the exact value of ϵ , but only on the $\epsilon \rightarrow 0$ limit.

While the exterior Schwarzschild solution is asymptotically recovered, which is an important feature by itself, it now connects with a novel interior solution. This new solution, which is completely different from Schwarzschild interior, is characterized by (i) No $t \leftrightarrow r$ signature flip, (ii) Drastically suppressed $e^{\nu(r), \lambda(r)}$, and (iii) Locally varying Newton constant. Altogether, the interior metric is well approximated by

$$ds_{in}^2 \simeq -\frac{\epsilon}{6} \left(\frac{r}{2GM} \right)^{\frac{6}{\epsilon}-4} dt^2 + \frac{6}{\epsilon} \left(\frac{r}{2GM} \right)^{\frac{6}{\epsilon}-6+2\epsilon} dr^2 + r^2 d\Omega^2, \quad (6)$$

with the built-in phase transition scale manifest. Such a short distance analytic behavior of the line element expresses the fact that all terms in the field equations which

are factorized by tiny $e^{\lambda(r)}$ are practically negligible now, including in particular the scalar potential terms. The corresponding locally varying Newton constant, defined as $\varphi^{-2}(r)$, being

$$G_{in}(r) \simeq G \left(\frac{r}{2GM} \right)^{2-\epsilon}. \quad (7)$$

Associated with any concentric inner sphere of a finite surface area $A(r) = 4\pi r^2$ is the invariant spatial volume

$$V(r) \simeq 4\pi \sqrt{\frac{2\epsilon}{3}} \left(\frac{r}{2GM} \right)^{\frac{3}{\epsilon}} (2GM)^3. \quad (8)$$

The fact that $V(r) \rightarrow 0$ for any $r < 2GM$, as $\epsilon \rightarrow 0$, is the reason why, unlike in any other macroscopic system, the black hole entropy cannot be proportional to its volume. This makes one wonder though how can physical degrees of freedom actually reside within the interior core.

To gain some more insight into the inner metric, one may verify the existence of a Kruskal-Szekeres structure, with the corresponding scale function being

$$S_\omega(r) \sim \left(\frac{r}{2GM} \right)^{\frac{6}{\epsilon}(1-4GM\omega)-4}, \quad (9)$$

or define proper a distance $\eta(r) \simeq \frac{2GM\sqrt{6\epsilon}}{3-2\epsilon+\epsilon^2} \left(\frac{r}{2GM} \right)^{\frac{3}{\epsilon}-2+\epsilon}$, and expand the inner metric up to $\mathcal{O}(\epsilon)$ pieces, to expose the Rindler structure of the \mathfrak{R}_2 sub-metric

$$ds_{in}^2 \simeq -\frac{\eta^2 dt^2}{16G^2 M^2} + d\eta^2 + 4G^2 M^2 \left(\frac{3\eta^2}{8G^2 M^2 \epsilon} \right)^{\frac{\epsilon}{3}} d\Omega^2. \quad (10)$$

The recovery the exact Hawking's imaginary time periodicity $\Delta\tau = 8\pi GM$ is regarded as the anchor connecting us to black hole thermodynamics. Notice, however, that unlike in the original Schwarzschild case, the Euclidean origin corresponds now to the center of spherical symmetry $r = 0$ rather than to $r = 2GM$.

Inside the inner core, the Kretschmann scalar is well approximated by

$$K = \mathcal{R}^{\mu\nu\lambda\sigma} \mathcal{R}_{\mu\nu\lambda\sigma} \simeq \frac{16\epsilon^2}{9\eta^4}, \quad (11)$$

(and $\mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} \simeq \frac{8\epsilon^2}{9\eta^4}$), but also contains the apparently negligible term $\frac{4}{r^4(\eta)}$. Note that K is suppressed by a factor $\frac{3}{4}\epsilon^2$ in comparison with its Schwarzschild analogue, and consequently, the singularity analysis bifurcates. (i) Clearly, for any finite η , as small as desired, the limit $\epsilon \rightarrow 0$ is *regular*. The Rindler sub-metric gets then multiplied by a 2-sphere of radius $2GM$, and consistently, the Kretschmann curvature approaches the Schwarzschild 'near horizon' value of $\frac{4}{(2GM)^4}$. However, (ii) For any finite ϵ , as small as desired, the limit $\eta \rightarrow 0$ is *singular*. Whereas the pseudo-horizon does provide some protection from the singularity (e.g. it takes an infinite amount

of time for light from the singularity to reach any external observer), an observer willing to wait long enough will see unbounded high curvature. Such a behavior is far worse than that of the Schwarzschild solution, and constitutes a severe problem. It may be that a more complicated Lagrangian could alleviate this behavior, or that quantum effects could cure it. At any rate, invoking Hawking's analysis, and taking into account the accompanying η^4 redshift of the radiation emanating from the inner core, an external observer would encounter the standard Hawking radiation.

At this point, we introduce the effective energy density $\rho(r)$ and the effective pressure $p(r)$. To clarify what exactly do we mean by the word effective, consider a physicist convinced that general relativity is the fundamental theory of gravity, and therefore totally unaware of its hereby advocated spontaneously generated nature. Such a physicist would re-arrange the underlying field equations into their basic Einstein form $\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G\mathcal{T}_{\mu\nu}$, moving all terms, save for the Einstein tensor itself, to the r.h.s., thereby constructing the effective energy/momentum tensor $\mathcal{T}_{\mu\nu}$. This does not change the fact that the physical metric remains a Jordan (not Einstein) frame metric, as dictated by the fact that in this frame, by construction, \mathcal{L}_m does not couple to the scalar field. Given the metric from eq.(6), one can easily verify that the effective energy/momentum tensor is

$$\mathcal{T}_{\nu}^{\mu} \simeq \frac{1}{r^2} \left(\frac{r}{2GM} \right)^{-\frac{6}{\epsilon}+6} \text{Diag}(-1, 1, 0, 0) . \quad (12)$$

Clearly, ρ and p diverge at the $\epsilon \rightarrow 0$ limit. However, they do so in such a way that $e^{\frac{1}{2}(\nu+\lambda)}\mathcal{T}_{\nu}^{\mu}$ actually converges, a crucial observation for the forthcoming discussion.

Consider now the proper mass $M(r)$ associated with some concentric sphere $\Omega(r)$. Recalling the recovery of the exterior Schwarzschild geometry for $\epsilon \rightarrow 0$, $M(r)$ must asymptotically approach M for all $r > 2GM$. Furthermore, $M(r)$ must be a finite monotonically increasing function of the circumferential radius r . Unfortunately, general relativity does not offer a unique definition for the term mass. Obviously, the naive mass formula $4\pi \int \rho r^2 dr$ will not do as it is non-covariant (and the integrand diverges), and the ADM mass only makes sense globally at asymptotically flat spatial infinity. At this stage, the tenable candidates are:

(i) The Komar mass [16] - Invoking Stoke's theorem and performing the angular integration, it acquires the form

$$M_K(r) = \frac{1}{2G} r^2 e^{\frac{1}{2}(\nu-\lambda)} \nu' . \quad (13)$$

This mass function requires the presence of a timelike Killing vector.

(ii) The material energy - Following Weinberg [17], this is the integration of the energy as measured in a locally inertial frame. The naive mass formula is then supple-

mented by the missing $\sqrt{-g_{tt} g_{rr}}$ factor to give

$$M_W(r) = \frac{1}{G} \int_0^r e^{\frac{1}{2}(\nu+\lambda)} (1 + e^{-\lambda}(\lambda' r - 1)) dr . \quad (14)$$

Given the metric eq.(6), and for $\epsilon \rightarrow 0$, we obtain for the entire core the mass function

$$M(r) = M \left(\frac{r}{2GM} \right)^2 . \quad (15)$$

The two mass definitions eqs.(13,14) are distinguished from each other only by their different $\mathcal{O}(\epsilon)$ corrections. The striking feature now is that unlike the Schwarzschild case, the present singularity does not induce any mass contribution whatsoever. The emerging Komar mass function appears to be well defined and non-singular!

The main question now is how will the black hole configuration change when supplementing M by a tiny amount δM ? First of all, as is well known, its overall radius would gain a $2G\delta M$ increase. But furthermore, appreciating the recovery of the Hawking temperature $T_H = (8\pi GM)^{-1}$ from the Rindler structure of the metric eq.(10), such a process would induce an entropy increase of $\delta S = \frac{\delta M}{T_H(M)}$, leading eventually to the famous Bekenstein-Hawking formula

$$S_{BH} = 4\pi GM^2 . \quad (16)$$

This, however, comes with no surprise. After all, as mentioned earlier, the interior Schwarzschild solution does not seem to play any role in the game. But in the present case, facing a self-similar profile transition into a novel interior core with no signature flip, it becomes meaningful and crucial to ask what portion of the above S_{BH} entropy, if any, which we denote by $S(r)$, is stored within an arbitrary inner sphere of a finite surface area $4\pi r^2$ hosting the mass $M(r)$?

The most general change in $M(r)$, namely

$$\delta M(r) = M(r) \left(2\frac{\delta r}{r} - \frac{\delta M}{M} \right) , \quad (17)$$

takes into account both the trigger shift $M \rightarrow M + \delta M$, as well as some suitable correction δr at any arbitrary inner sphere radius r . For the special case $r = 2GM$, for example, we have $\delta r/r = \delta M/M$. The crucial point to notice now is that, for $\epsilon \rightarrow 0$, the volume $V(r)$ is solely a function of the ratio r/M , with M being the only length scale in the inner metric. In other words, the new $(M + \delta M)$ -configuration held in thermal equilibrium is nothing but a linearly stretched version of the old M -configuration. Thus, conveniently choosing δr , such that $\delta(r/M) = 0$, for all $r < 2GM$, will assure $\Delta V = 0$. Implementing this physical choice, the first law of thermodynamics (with the Hawking temperature T_H at infinity) can be put to work in the following form

$$\frac{\delta S(r)}{8\pi GM} = \delta M(r) = \frac{r\delta r}{4G^2 M} . \quad (18)$$

Taking advantage of the M cancelation, we finally arrive at our main result

$$S(r) = \frac{\pi r^2}{G} = S_{BH} \left(\frac{r}{2GM} \right)^2. \quad (19)$$

The emerging entropy packing profile turns out to be (i) Locally holographic, i.e. exhibits proportionality to $A(r)$ for every $r \leq 2GM$, and (ii) M -independent, and thus universal. The overall picture is then of an onion-like model. The entropy of any inner sphere is maximally packed, and unaffected by the outer layers. Any additional entropy is maximally packed on its own external layer, with M as well as $M(r)$ being adjusted accordingly. In some sense, each layer acts as an event horizon [18], with the local light cone $\frac{dr}{dt} \simeq \pm \epsilon \left(\frac{r}{2GM} \right)$ getting closed at the $\epsilon \rightarrow 0$ limit for every r in the core. Notice that the derived entropy function eq.(19) is in full agreement with the 't Hooft-Susskind holographic bound [7, 8]. In fact, it locally saturates this bound!

An interesting point has to do with the entropy to energy ratio, which in our case reads

$$\frac{S(r)}{M(r)} = \frac{S_{BH}}{M} = 4\pi GM \geq 2\pi r. \quad (20)$$

This is in apparent violation of Bekenstein's universal entropy bound [19]. The reason seems to be the following. Whereas the entropy $S(r)$ of some inner sphere is universal, the associated mass $M(r) \sim M^{-1}$ is affected by the total mass of the system. Admittedly, Bekenstein's universal bound is relevant [20] only for weakly self gravitating isolated physical systems (and for these it is a much stronger bound than the holographic one), but it may still be applicable here provided the radius appearing in universal bound is of the entire system rather than of a subsystem.

Altogether, the emerging local realization of the holographic principle, although highly unconventional, is very pleasing. The new entropy packing profile sheds new light on the way information is stored within a black hole, and this is achieved without invoking string theory or the AdS/CFT correspondence. Rather than envision bits of information evenly spread solely on the horizon surface or in its vicinity, a bit per Planck area, they are now universally and holographically spread in the whole black hole interior. Our results stem from the spontaneous generation of general relativity, and are insensitive to the explicit form of the scalar potential. Crucial to our analysis is the identification of the tenable mass definition, with the Komar mass and the material energy being the leading candidates. This choice is a posteriori justified once the holographic structure is fully revealed. Furthermore, the non-singular mass distribution eq.(15) appears to be intimately related to, and thus as fundamental as, the entropy distribution eq.(19). Exactly the same structure is expected to hold once the cosmological constant and/or the electric charge enter the game.

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